Theorem 2

This is an expanded version of the proof given in *Real Analysis* by John Howie (pp 34) and is intended to make the proof more accessible to students learning this subject. Any mistakes are the fault of this website only.

Theorems stated without proof

Theorem 1 :

Let a_n be a convergent sequence with limit α. Then $|a_n|$ is convergent with limit $|\alpha|$.

See [1] pp 30 for proof.

Theorem 2 :

Every convergent sequence is bounded

Proof:

Let a_n be a sequence with limit α. Then by Theorem 1 we know that $|a_n|$ has limit $|\alpha|$. Thus we can say;

$$\begin{aligned} &\left||a_n|-|\alpha|\right| < \varepsilon \qquad n > N \\ &-\varepsilon < |a_n|-|\alpha| < \varepsilon \\ &-\varepsilon + |\alpha| < |a_n| < \varepsilon + |\alpha| \end{aligned} \tag{1}$$

Now the limit α of this sequence may or may not be the maximum or minimum of the sequence. By (1) we know that for $i = 1,.....N$ $|a_n|$ may take any value but for $i = N+1, N+2,.....,\infty$ $|a_n| < \varepsilon + |\alpha|$. Thus we can now state that using (1), and for all $n \geq 1$,

$$|a_n| \leq \max\{|a_1|,|a_2|,......,|a_N|,\varepsilon + |\alpha|\}$$

showing that a_n is bounded.

QED.

References

[1] Howie, J.M, (2006), Springer *Undergraduate Mathematics Series: Real Analysis*, Springer, London.

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