This is an expanded version of the proof given in *Elementary Linear Algebra* by Howard Anton and Chris Rorres, pp 192, (see [1]) and is intended to make the proof more accessible to students learning this subject. Any mistakes are the fault of this website only.

Note: A * at the start of a line shows where the original proof has been expanded/modified.

A note on linear transformations

This section is a summary of pages 174-175 of [1].

The following shows the main type of transformations dealt with in linear algebra;

$$\begin{aligned} y_1 &= f_1(x_1, x_2, ..., x_n) \\ y_2 &= f_2(x_1, x_2, ..., x_n) \\ &\vdots \\ y_m &= f_m(x_1, x_2, ..., x_n) \end{aligned} \qquad (1)$$

The domain of each of the m function takes n input values, so as a whole set, the domain of the functions is in R^n whilst the range or codomain of the functions is in R^m . This transformation is denoted $T : R^n \to R^m$. The following shows a particular form of (1);

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \end{aligned} \qquad (2)$$

* These are called linear functions. The reason they are linear is that say x_1 increases from x_1 to $x_1^{'}$. Then both values are simply weighted by a coefficient and so $a_{11}x_1$ increases or decreases proportionally with x_1 . This is in contrast to say e^{x_1} which is a non-linear function. If A is an m x n matrix and **x** is a n x 1vector then (2) can be written;

y=A**x** (3)

And so a linear transformation T is defined by;

$T(\mathbf{x}) = A\mathbf{x}$ (4)

Theorems / Results stated without proof

(i): $T(v_1+v_2+...+v_k)=T(v_1)+T(v_2)+...T(v_k)$ where $v_1, v_2, ...v_k \in R^n$

Theorem 4: Properties of linear transformations

*A transformation $T: R^n \to R^m$ is linear if and only if the following relationships hold for all vectors **u** and **v** in R^n and every scalar c.*
a) $T(u+v)=T(u)+T(v)$ *b)* $T(cu)=cT(u)$

Proof;

If we assume that a) and b) hold for a transformation T then we can show that T is linear if $T(\mathbf{x})=A\mathbf{x}$ for all $\mathbf{x} \in R^n$.

Let;

$$\mathbf{e_1}=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \mathbf{e_2}=\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \ldots\ldots, \mathbf{e_n}=\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \quad (5)$$

Now if A is an m x n matrix and **x** is a n x 1vector then $A\mathbf{x}$ can be written;

$$\begin{bmatrix} a_{11}x_1+a_{12}x_2+...+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+...+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+...+a_{mn}x_n \end{bmatrix} = x_1\mathbf{a_1}+x_2\mathbf{a_2}+\ldots\ldots+x_n\mathbf{a_n} \quad (6)$$

*

where the **a's** are the column vectors of *A*. Now if we apply the transformation T to the column vectors in (5), and let the columns of *A* be these transformed column vectors, then (6) becomes;

$$\begin{aligned} A\mathbf{x} &= x_1T(\mathbf{e_1})+x_2T(\mathbf{e_2})+\ldots\ldots+x_nT(\mathbf{e_n}) \\ &= T(x_1\mathbf{e_1})+T(x_2\mathbf{e_2})+ \quad +T(x_n\mathbf{e_n}) && \text{(property b)} \\ &= T(x_1\mathbf{e_1}+x_2\mathbf{e_2}+ \quad x_n\mathbf{e_n}+) && \text{(property a extended to } n \text{ terms)} \quad (7) \end{aligned}$$

* Now;

$$x_1\mathbf{e_1} + x_2\mathbf{e_2} + \dots + x_n\mathbf{e_n} = \begin{bmatrix} e_{11}x_1 + e_{12}x_2 + \dots + e_{1n}x_n \\ e_{21}x_1 + e_{22}x_2 + \dots + e_{2n}x_n \\ \vdots \\ e_{m1}x_1 + e_{m2}x_2 + \dots + e_{mn}x_n \end{bmatrix}$$

$$* = \begin{bmatrix} (1)x_1 + 0 + \dots + 0 \\ 0 + (1)x_2 + \dots + 0 \\ \vdots \\ 0 + 0 + \dots + (1)x_n \end{bmatrix}$$

$$* = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}$$

So (7) can be written;

$A\mathbf{x} = T(\mathbf{x})$.

Proving the theorem the other way, if we assume that T is a linear transformation and A is the matrix for T then;

$T(u + v) = A(u + v) = Au + Av = T(u) + T(v)$ and;

$T(cu) = A(cu) = c(Au) = cT(u)$

QED

<u>References</u>

[1] Anton, H and Rorres, C, (2000), *Elementary Linear Algebra*, Eigth Edition, John Wiley & Sons.

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