This is an expanded version of the proof given in *Real Analysis* by John Howie, pp 34, (see [1]) and is intended to make the proof more accessible to students learning this subject. Any mistakes are the fault of this website only.

Note: A * at the start of a line shows where the original proof has been expanded/modified.

Theorems stated without proof

Theorem 1 :

Let a_n be a convergent sequence with limit α. Then $|a_n|$ is convergent with limit $|\alpha|$.

See [1] pp 30 for proof.

Theorem 2 :

Every convergent sequence is bounded

http://qedstats.wordpress.com/2008/07/01/theorem-2/

Theorem 3 :

Results involving two convergent sequences.

Let a_n and b_n be two sequences with limits α and β respectively then;

(i) $(-a_n) \to -\alpha$
(ii) $(a_n + b_n) \to \alpha + \beta$
(iii) $(a_n - b_n) \to \alpha - \beta$
(iv) $(a_n b_n) \to \alpha\beta$
(v) $(ka_n) \to k\alpha$ for every constant k
(vi) $\left(\frac{1}{b_n}\right) \to \left(\frac{1}{\beta}\right)$ if $b_n \neq 0$ for all n and $\beta \neq 0$
(vii) $\left(\frac{a_n}{b_n}\right) \to \left(\frac{\alpha}{\beta}\right)$ if $b_n \neq 0$ for all n and $\beta \neq 0$

Proof for (iv):

Since a_n and b_n are both convergent sequences then by Theorem 2 they both have upper bounds A and B (both positive) for all $n \geq 1$. Now suppose that $|\beta| > B$.

* If $|\beta|$ is very close to B then $|\beta| - B$ will be very small. Thus $|\beta| - B = \varepsilon$ say.

* Using this and Theorem 1 we have;

* $\left||b_n| - |\beta|\right| < \varepsilon = |\beta| - B$ for all $n > N$ (Note that this still holds if $|\beta|$ is not close to the upper bound B.)

* $-|\beta| + B < |\beta| - |b_n| < |\beta| - B$
* $-|b_n| < -B$
* and so $|b_n| > B$
*. But this contradicts B as an upper bound and so $|\beta| \leq B$ for all $n > N$. This result is the same for α. Therefore;

$$|\beta| \leq B \text{ for all } n > N_1 \text{ and } |\alpha| \leq A \text{ for all } n > N_2 \tag{1}$$

Now let $\varepsilon > 0$, and because A and B are both positive;

$$|a_n - \alpha| < \frac{\varepsilon}{2B} \quad n > N_1 \qquad\qquad |b_n - \beta| < \frac{\varepsilon}{2A} \quad n > N_2 \tag{2}$$

* i.e. ε is an arbitrarily small number and so is $\frac{\varepsilon}{2B}$.

Now;

$$|a_n b_n - \alpha\beta| = |a_n(b_n - \beta) + \beta(a_n - \alpha)| \tag{3}$$

* i.e inserting $-a_n\beta + a_n\beta$ into the LHS and factorizing. Now if all of the bracketed terms
* and non-bracketed terms on the RHS are positive then we have;

* $|a_n(b_n - \beta) + \beta(a_n - \alpha)| = (|a_n|)(|b_n - \beta|) + (|\beta|)(|a_n - \alpha|)$

* But if any of the two products on the LHS are less than zero then the LHS is less than the RHS and so (3) can be written;

$$\begin{aligned} |a_n(b_n - \beta) + \beta(a_n - \alpha)| &\leq (|a_n|)(|b_n - \beta|) + (|\beta|)(|a_n - \alpha|) \\ &\leq A(|b_n - \beta|) + B(|a_n - \alpha|) \qquad \text{(using (1))} \end{aligned}$$

and for all $n > \max\{N_1, N_2\}$;

$$< A\left(\frac{\varepsilon}{2A}\right) + B\left(\frac{\varepsilon}{2B}\right) \qquad \text{(using (2))}$$

$$= \left(\frac{\varepsilon}{2}\right) + \left(\frac{\varepsilon}{2}\right) = \varepsilon$$

Thus;

$\lvert a_n b_n - \alpha\beta \rvert < \varepsilon$ for all $n > \max\{N_1, N_2\}$

and so $a_n b_n \to \alpha\beta$ QED.

Proof for (vi):

Since $\beta \neq 0$ and $b_n \to \beta$ then;

$$\lvert b_n - \beta \rvert < \varepsilon = \frac{\lvert\beta\rvert}{2} \qquad \text{for all } n > N_1. \qquad (4)$$

* i.e. Since $b_n \to \beta$ then $\lvert b_n - \beta \rvert$ can be made arbitrarily small and can therefore be made
* smaller than $\frac{\lvert\beta\rvert}{2}$.

* The table below compares three expressions under different conditions (the conditions
* are given in the first two columns).

		(i) $\lvert\beta\rvert - \lvert b_n \rvert$	(ii) $\left\lvert \lvert b_n\rvert - \lvert\beta\rvert \right\rvert$	(iii) $\lvert b_n - \beta \rvert$	
a) $b_n > 0, \beta > 0$	$b_n > \beta$	$\beta - b_n (< 0)$	$\lvert b_n - \beta \rvert = b_n - \beta (> 0)$	$\lvert b_n - \beta \rvert = b_n - \beta (> 0)$	(i)<(ii)=(iii)
	$\beta > b_n$	$\beta - b_n (> 0)$	$\lvert b_n - \beta \rvert = \beta - b_n (> 0)$	$\lvert b_n - \beta \rvert = \beta - b_n (> 0)$	(i)=(ii)=(iii)
	$\beta = b_n$	0	0	0	(i)=(ii)=(iii)
b) $b_n < 0, \beta > 0$	$\beta > b_n$	$\beta - \lvert b_n \rvert$ (<0,>0)	$\left\lvert \lvert b_n \rvert - \beta \right\rvert (> 0)$	$\lvert b_n - \beta \rvert (> 0)$	(i)$\leq$(ii)<(iii)
c) $b_n > 0, \beta < 0$	$b_n > \beta$	$\lvert\beta\rvert - b_n (< 0, > 0)$	$\left\lvert b_n - \lvert\beta\rvert \right\rvert (> 0)$	$\lvert b_n - \beta \rvert (> 0)$	(i)$\leq$(ii)<(iii)
d) $b_n < 0, \beta < 0$	$b_n > \beta$	$\lvert\beta\rvert - \lvert b_n \rvert (> 0)$	$\lvert\beta\rvert - \lvert b_n \rvert (> 0)$	$\lvert\beta\rvert - \lvert b_n \rvert (> 0)$	(i)=(ii)=(iii)
	$\beta > b_n$	$\lvert\beta\rvert - \lvert b_n \rvert (< 0)$	$\lvert b_n \rvert - \lvert\beta\rvert (> 0)$	$\lvert b_n \rvert - \lvert\beta\rvert (> 0)$	(i)<(ii)=(iii)
	$\beta = b_n$	0	0	0	(i)=(ii)=(iii)

From the above table and (4) we can now write;

$$|\beta| - |b_n| \leq \Big|\, |b_n| - |\beta| \,\Big| \leq |b_n - \beta| < \varepsilon = \frac{|\beta|}{2}$$

$$|\beta| - |b_n| < \frac{|\beta|}{2}$$

$$-|b_n| < -\frac{|\beta|}{2}$$

$$|b_n| > \frac{|\beta|}{2}$$

$$\frac{1}{|b_n|} < \frac{2}{|\beta|} \tag{5}$$

Now since $b_n \to \beta$ we can say;

$$|b_n - \beta| < \frac{\varepsilon|\beta|^2}{2} \qquad n > N_2$$

i.e. b_n can be made arbitrarily small. Now multiplying both sides by an expression;

$$\frac{2|b_n - \beta|}{|\beta|^2} = \frac{2}{|\beta|}\frac{|b_n - \beta|}{|\beta|} < \varepsilon$$

and so from (5);

$$\frac{1}{|b_n|}\frac{|b_n - \beta|}{|\beta|} < \frac{2}{|\beta|}\frac{|b_n - \beta|}{|\beta|} < \varepsilon$$

$$\frac{|b_n - \beta|}{|b_n\beta|} < \frac{2|b_n - \beta|}{|\beta|^2} < \varepsilon$$

$$\left|\frac{1}{\beta} - \frac{1}{b_n}\right| = \frac{|b_n - \beta|}{|b_n\beta|} < \frac{2|b_n - \beta|}{|\beta|^2} < \varepsilon$$

$$\left|-\left[\frac{1}{\beta} - \frac{1}{b_n}\right]\right| < \varepsilon$$

$$\left|\frac{1}{b_n} - \frac{1}{\beta}\right| < \varepsilon$$

and so $\dfrac{1}{b_n} \to \dfrac{1}{\beta}$

QED.

References

[1] Howie, J.M, (2006), Springer *Undergraduate Mathematics Series: Real Analysis*, Springer, London.